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DETERMINING GENERAL RELATIVITY-BASED STATIC SPHERICALLY SYMMETRIC SOLUTIONS TO EINSTEIN'S EQUATION

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ABSTRACT

Exact solutions of Einstein's field equations in closed analytic form are difficult to obtain, on account of complicated and nonlinearity of the equations. In the study of generating static spherically symmetric solutions in general relativity, researchers often manipulate the field equations to simplify the analysis. This paper outlines various methodologies for deriving static spherically symmetric solutions based on existing solutions. On use of the techniques, we have generated a 1- parameter family of two new solutions.

KEYWORDS: *Constant density, Perfect fluid, Spherically symmetric, Pressure,1- Parameter*

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1. INTRODUCTION

Developing static, spherically symmetric solutions to the general relativity equations put forth by Einstein is a fundamental aim of theoretical physics. These answers are essential for comprehending the behavior of matter and energy in the presence of gravitational fields. In 1918, Schwarzschild discovered the first solution of this type with constant density **[1]**. Since then, other static spherically symmetrical solutions of fluids have been identified **[2], [3]**. A mechanism for classifying the set of ideal fluid solutions is provided by numerous solutions generating approaches **[4], [5]**. Additionally, these methods might be applied to generate new responses by preexisting ones, and they depend on the Buchdahl limit **[6]**. Multiple investigations have concentrated on acquiring precise solutions of Einstein's equations for systems that are both static and spherically symmetric. Golovnev provides a pedagogical overview of static spherically symmetric solutions in novel models of general relativity **[7]**. The focus is on demonstrating the ease and usefulness of getting these solutions. Kiselev revealed novel static spherically symmetric precise solutions to Einstein's equations, include archetypal matter enclosing a black hole **[8]**. The significance of these solutions resides in their capacity to characterize the features of matter in the neighborhood of black holes. Ali et al. brought attention to the importance of static spherically symmetric solutions in general relativity **[9]**. Birkhoff demonstrated that these solutions are both asymptotically flat and static. Apart from the implications in matter field behavior and cosmology,static spherically symmetric solutions are also important.A theorem presented by Salgado provides a method for employing matter fields to get accurate solutions to the Einstein field equations **[10]**.

These spherically symmetric, static solutions are essential to understanding the behavior of matter when gravitational fields are present. Static, spherically symmetric solutions to Einstein's general relativity equations are immensely vital to theoretical physicists. It greatly shapes our knowledge of the fundamental properties of matter, energy, and gravitational fields. The outcomes could disclose critical new information concerning the dynamics of matter fields, the mechanism of black holes, and the potential implications of cosmological theories **[11], [12], [13], [14]**.

1. Governing Field Equations

In curvature coordinates, here is an expression for the metric system describes the properties of space and time in a static and spherically symmetric manner:

$$.d\ ^2 = -e^{2\varphi(r)}d\ ^2 + e^{2\Lambda(r)}d\ ^2 + r^2(d\ ^2 + si\ ^2\theta d\ ^2) \quad (1)$$

The nonzero elements of Einstein's tensor are obtained for the metric (1) by,

$$G_0 = \frac{1}{r^2}e^{2\varphi}\frac{d}{dr}r(1-e^{-2\Lambda}) \quad (2)$$

$$G_r = -\frac{1}{r^2}(1-e^{-2\Lambda}) + \frac{2}{r}e^{-2\Lambda}\varphi' \quad (3)$$

$$G_{\theta\theta} = r^2e^{-2\Lambda}\left\{\varphi'' + (\varphi')^2 + \frac{\varphi'}{r} - \left(\varphi' + \frac{1}{r}\right)\Lambda'\right. \quad (4)$$

$$G_{\varphi} = G_{\theta 6}\sin^2\theta \quad (5)$$

Here, we use primes to represent derivatives with respect to r. For the purpose of this analysis, we will assume that the fluid sphere is perfect and focus on its elemental content. The energy-momentum tensor (T_{μ}) in this scenario consists of non-zero elements which are given by,

$$T_0 = \rho e^{2\varphi} \quad (6)$$

$$T_r = pe^{2\Lambda} \quad (7)$$

$$T_{\theta\theta} = pr^2 \quad (8)$$

$$T_{\varphi\varphi} = T_{\theta\theta}\sin^2\theta \quad (9)$$

where the density and pressure can be represented in the fluid sphere by, $\rho(r)$ and $p(r)$, respectively. we find the independent equations by putting (2)-(9) in the Einstein's equation $G_{\mu} = 8\pi T_{\mu}$, so we get,

$$8\pi = \frac{1}{r^2}e^{2\varphi}\frac{d}{d}r(1-e^{-2\Lambda}) \quad (10)$$

$$8\pi p = -\frac{1}{r^2}(1-e^{-2\Lambda}) + \frac{2}{r}e^{-2\Lambda}\varphi' \quad (11)$$

$$8\pi = r^2e^{-2\Lambda}\{\varphi'' + (\varphi')^2 + \frac{\varphi'}{r} - (\varphi' + \frac{1}{r})\Lambda' \quad (12)$$

from (11) and (12) we obtain the equation,

$$\frac{d}{d}\{e^{-2\Lambda}\}+\frac{2\{r^2\varphi''+(r\varphi')^2-r\varphi'-1\}}{r(r\varphi'+1)}e^{-2\Lambda}=-\frac{2}{r(r\varphi'+1)} \quad (13)$$

If we let $\{e^{-2\Lambda}=1-\frac{2m}{r}\}$ and $\{u=\frac{m}{r}\}$, then (13) reduces to,

$$u'+\frac{2\{r^2\varphi''+(r\varphi')^2-r\varphi'-1\}}{r(r\varphi'+1)}u=\frac{r^2\varphi''+(r\varphi')^2-r\varphi'}{r(r\varphi'+1)} \quad (14)$$

2. Techniques for generating solutions

Some solutions generation methods are discussed in [1] and [2]. These methods help a known solution to be converted into a new one. Let, N(r) = $e^{\varphi(r)}$ and G(r) = $e^{-\Lambda(r)}$then one can employ this approach. Equation (14) then simplifies to,

$$G'+\frac{2(r^2N''-rN'-N)}{r(rN'+N)}G+\frac{2N}{r(rN'+N)}=0 \quad (15)$$

equations (15) can be written as,

$$(2r^2G)N''+(r^2G'-2rG)N'+(rG'-2G+2)N=0 \quad (16)$$

3. Techniques for developing solutions can be characterized as follows

(A) In the beginning, we assume with a solution $(N_0(r),G_0(r))$ i.e. we suppose that the equation,

$$G_0{}'+\frac{2(r^2N_0{}''-rN_0{}'-N_0)}{r(rN_0{}'+N_0)}G_0+\frac{2N_0}{r(rN_0{}'+N_0)}=0 \quad (17)$$

is fulfilled. Let us now claim that $(N_0(r),G_1(r))$, where $G_1(r)=G_0(r)+k\Delta_0(r)$ is a solution. Putting $(N(r),G(r))=(N_0(r),G_1(r))$ in (15) and using (17) we get,

$$\Delta_0{}'+\frac{2(r^2N_0{}''-rN_0{}'-N_0)}{r(rN_0{}'+N_0)}\Delta_0=0 \quad (18)$$

solution of (18) is given by,

$$\Delta_0(r)=\frac{r^2}{\{(rN_0)'\}^2}\exp\int\frac{4N_0{}'}{(rN_0)'}dr \quad (19)$$

Therefore, we conclude that of $(N_0(r),G_0(r))$is a known solution and of $\Delta_0(r)$ is given by (19), then,

$(N_0(r),G_0(r)+k\Delta_0(r))$is another possibility.

The outcome could be seen as a modification,

$$T_1:(N_0(r),G_0(r))\rightarrow(N_0(r),G_0(r)+k\Delta_0(r))$$

In the set of solutions of equation (15).

(B) Considering $(N_0(r),G_0(r))$ as a solution to equation (16), meaning we enable,

$$(2r^2G_0)N_0{}''+(r^2G_0{}'-2rG_0)N_0{}'+(rG_0{}'-2G_0+2)N_0=0 \quad (20)$$

and demand that $(N_1(r),G_0(r))$ where, $N_1(r)=N_0(r)Z_0(r)$ be a solution. Putting $(N_1(r),G_0(r))$ in equation (16) and using (20) we get the equation,

$$z_0'' + \frac{r^2N_0G_0' - 4r^2N_0'G_0 - 2rN_0G_0}{2r^2N_0G_0}z_0' = 0 \qquad (21)$$

This is an equation of first order for linear differentials in z_0'. The solution of equation (21) is obtained by,

$$z_0 = c_1 + c_2\int \frac{rdr}{N_0^{\,2}\sqrt{G_0}} \qquad (22)$$

which depend on the starting solution (N_0, G_0). Therefore, we conclude that if (N_0, G_0) is a solution, then (N_0z_0, G_0) where,z_0 is given by (22), is also a solution. The result can be observed as the conversion,

$$T_2: (N_0, G_0) \rightarrow (N_0z_0, G_0)$$

in the set of solutions of equation (21).

4. Generation of new solution

5.1 Application of (1) to Minkowski metric:

Here we have developed a class of new approaches with the method mentioned in (1). For this we apply transformation $T_1$to the Minkowski metric and the solution given by,

$$(N_0, G_0) = (1, 1 + ar^2)$$

After putting the above value of in equation (19), we get the value of $\Delta_0(r)$,so we can write,

$$\Delta_0(r) = r^2 \qquad (23)$$

and $$G_1(r) = G_0(r) + k\Delta_0(r)$$

$$G_1(r) = 1 + kr^2$$

Then we get a new solution,

$$d\ ^2 = -d\ ^2 + \frac{d\ ^2}{G_0 + k\Delta_0} + r^2d\Omega^2 \qquad (24)$$

$$= -d\ ^2 + \frac{d\ ^2}{1 + kr^2} + r^2d\Omega^2 \qquad (25)$$

Metric (25) describe is a new parameter family, characterized by k, of symmetrical, static, perfectly spherical fluid (ssspf) solutions.

5.2 Properties of the solutions:

Now comparing equation (1), (15), (25), and let that k=a for simplicity. Then, we get,

$$N^2(r) = e^{2\varphi(r)} = 1 \qquad (26)$$

and

$$.\ G^2 = e^{-2\Lambda(r)} = 1 + ar^2 \qquad (27)$$

Now, calculating equation (10) and (11), putting the value of equation (26) and (27), we get,

$$\rho(r) = -\frac{3a}{8\pi} \quad (28)$$

$$p(r) = \frac{a}{8\pi} \quad (29)$$

it's clearly evident that both $p(r)$ and $\rho(r)$ are constant for the schwarzchild interior solution. Moreover, the central density, $\rho_0(r)$ and central pressure $p_0(r)$ will also be a constant.

5.3 Application of (2) to Einstein static metric:

Here we have developed a class of new approaches with the method mentioned in (2). For this we apply transformation $T_2$to the Einstein static metric and the solution given by,

$$(N_0, G_0) = (1, 1 + kr^2)$$

After putting the above value of in equation (22), we get the value of $z_0(r)$, so we can write,

$$z_0(r) = c_1 + c_2 \int \frac{r}{N_0{}^2\sqrt{G_0}} \quad (30)$$

$$z_0(r) = c_1 + c_2 \int \frac{rdr}{\sqrt{1 + kr^2}} \quad (31)$$

After calculating equation (31), we get the solution,

$$z_0 = c_1 + c_2{}'\sqrt{1 + kr^2} \quad (32)$$

Then we get a new solution,

$$ds^2 = -\left(c_1 + c_2{}'\sqrt{1 + kr^2}\right)^2 dt^2 + \frac{dr^2}{1 + kr^2} + r^2 d\Omega^2 \quad (33)$$

$$= -c_2{}'^2\left(\frac{c_1}{c_2{}'} + \sqrt{1 + kr^2}\right)^2 dt^2 + \frac{dr^2}{1 + kr^2} + r^2 d\Omega^2$$

$$= -A^2\left(l + \sqrt{1 + kr^2}\right)^2 dt^2 + \frac{dr^2}{1 + kr^2} + r^2 d\Omega^2 \quad (34)$$

If we let that term, $\frac{c_1}{c_2{}'} = l$, $c_2{}' = A$,it will be more convenient to simplify this equation.Hence, Equation (32) is a new parameter family also, characterized by k, in ssspf.

5.4 Properties of the solutions:

Now comparing equation (1), (15), (33), and let that k=a for simplicity. Then, we get,

$$.N^2(r) = e^{2\varphi(r)} = A^2\left(l + \sqrt{1 + kr^2}\right)^2 \quad (35)$$

And

$$.G^2(r) = e^{-2\Lambda(r)} = \frac{1}{1+ar^2} \quad (36)$$

Now, calculating equation (10) and (11), putting the value of equation (35) and (36), we get,

$$\rho(r) = -\frac{3a}{8\pi} \tag{37}$$

$$p(r) = \frac{a}{8\pi}\left[1 + \frac{2\sqrt{1+ar^2}}{l+\sqrt{1+ar^2}}\right] \tag{38}$$

Putting r=0 enables us to get central density and pressure. so, we get $p_c$and $\rho_c$as follow,

$$\rho_c = -\frac{3a}{8\pi} \tag{39}$$

$$p_c = \frac{a}{8\pi}\left[1 + \frac{2}{l+1}\right] \tag{40}$$

Assuming that the fluid sphere's surface be represented by r=R such that $p(R) = 0$, from equation (38) we get,

$$0 = \frac{a}{8\pi}\left[1 + \frac{2\sqrt{1+aR^2}}{l+\sqrt{1+aR^2}}\right]$$

$$= \frac{a}{8\pi}\left[\frac{l+3\sqrt{1+aR^2}}{l+\sqrt{1+aR^2}}\right] \tag{41}$$

On the surface, p(R)=0, if we use, $G^2 = 1 - \frac{2m(r)}{r}$, we can write,$\frac{2M}{R} = aR^2$ and let l=1. So,after using the value, we get,

$$\frac{l+3\sqrt{1+aR^2}}{l+\sqrt{1+aR^2}} = 0 \tag{42}$$

$$\text{or,}\quad \frac{M}{R} = \frac{1}{2}\left(1 - \frac{l^2}{9}\right) \tag{43}$$

Now, from equation (36) we can calculate the value of m(r), so we can write,

$$G^2(r) = e^{2\Lambda(r)} = 1 - \frac{2m(r)}{r} = 1 + ar^2$$

$$\therefore m(r) = -\frac{ar^3}{2} \tag{44}$$

If we putting r=R in equation (43), The mass of the fluid sphere, M=m(R), can be determined. then, we get,

$$M(r) = -\frac{aR^3}{2} \tag{45}$$

Now,

$$e^{2\varphi(r)} = 1 - \frac{2M}{R} = A^2\left(l + \sqrt{1+kr^2}\right)^2 \tag{46}$$

$$\therefore A = \frac{\sqrt{1 - \frac{2M}{R}}}{l+\sqrt{1+aR^2}} \tag{47}$$

So, putting the value of A in equation (35), we get the value,

$$e^{2\varphi(r)} = A^2\left(l+\sqrt{1+kr^2}\right)^2$$

$$= \left(1-\frac{2M}{R}\right)\left(\frac{l+\sqrt{1+kr^2}}{l+\sqrt{1+kR^2}}\right) \quad (48)$$

Specific case, l=1, -1, 2, -2:

If we putting the value l=1 or -1,2 or -2 in equation (43), we find that the value satisfied our theory and holds Buchdahl conditions [7].

$$\therefore \frac{M}{R} = \frac{4}{9} \quad (49)$$

For, l=1, -1 the equation (43) shows,$\frac{M}{R}$ is equal to $\frac{4}{9}$.

$$\therefore \frac{M}{R} < \frac{5}{18} < \frac{4}{9} \quad (50)$$

For l=2, -2, the same equation holds the Buchdahl conditions.

5. CONCLUSION

We have provided the procedures which can be able to generate all types of static spherically symmetric ideal fluid solutions of Einstein's equations, that derive from the choice of a generating function. In order to demonstrate mathematically that the ratio of M/R is not only always less than 4/9, but it can also be equal to the value, we are able to illustrate this by the utilization of this technique. In addition, the particular instance indicates that it is also capable of upholding the Buchdahl criteria.

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